Int. J. Solids Structures, Vol. 17, pp. 411–420, 1981
Printed in Great Britain.

0020-7683/81/040411-10$02.00/0

STRESS DIFFUSION IN NON-LINEAR INTERPENETRATING BARS

PIERO VILLAGGIO
Istituto di Scienza delle Costruzioni dell'Università di Pisa, 56100 Pisa, Italy

(*Received* 4 *January* 1980; *in revised form* 23 *July* 1980)

Abstract—We study the stress distribution in a simple bar constituted by two isotropic, homogeneous, linear elastic constituents. We discuss three constitutive assumptions on the interacting force: namely, it is purely elastic; it is elastic-perfectly plastic; it has a brittle response.

1. INTRODUCTION

In recent years a certain interest towards the theory of interacting continua has been registered. For instance, Green and Naghdi[1] derived the constitutive equations for a mixture of two Newtonian compressible fluids by using the general principles of continuum mechanics; Crochet and Naghdi[2] considered the non-linear constitutive equations for the flow of a fluid through an elastic solid. Later, Green and Steel[3] gave the explicit form of the constitutive equations for a mixture of two non-linear elastic solids. Aron[4] studied the restrictions imposed on the linear constitutive equations of a mixture of two elastic solids by the requirement that the corresponding boundary-value problems are well posed.

A different approach was employed by Tiersten and Jahanmir[5], who regarded the mixture of two elastic solids as a composite of interpenetrating continua, in which the motion of a particle of the combined continuum could be finite while the relative motion of each of the constituents was infinitesimal.

It must be observed, however, that the linear constitutive equations for a two-constituent isotropic composite derived by Green and Steel[3] and those obtained by Tiersten and Jahanmir[5] are not completely equivalent.

Interacting continua can be used as a macroscopic model of composite material, that is a material composed by a number of distinct constituents with different physical properties. The macroscopic equations clearly do not consider the detailed motion of each individual component. Nevertheless they give certain types of information, as, for instance, the actual bond force between the constituents.

In this paper I consider a two-constituent composite material which behaves in the following way. Each constituent is linear elastic, but the bond force may either be elastic-perfectly plastic or elastic-perfectly brittle. Constitutive equations of this kind are appropriate to describe the response of many composite materials under high stresses. While the individual components of the solid mixture are relatively strong, the connections between different constituents are relatively weak.

In order to obtain simple, but definite results, I only consider the distributions of stresses and displacements in a rectilinear bar of a two-constituent composite material under simple traction. Equations similar to the ones given below were derived by Jahanmir and Tiersten[6] in studying the one dimensional load transfer in fiber reinforced composite materials. It is possible in this case to get exact solutions for the considered boundary value problems. The region where the interacting forces change their nature can also be determined.

A rather unexpected result is that the macroscopic relationship between the tensile force and the corresponding displacement reproduces the behaviour of the elastic-plastically hardening material.

2. BASIC EQUATIONS

The bar under consideration consists of two distinct isotropic linear elastic continua. Initially, both continua occupy the same region and, hence, have the same material coordinate X ranging over a closed, finite interval $0 \leq X \leq l$. After deformation, a generic particle of the first continuum takes the position,

$$x_1 = x_1(X), \tag{2.1}$$

and a generic particle of the second continuum takes the position

$$x_2 = x_2(X). \tag{2.2}$$

We assume that both these mappings are one-to-one and differentiable as often as required. Let us introduce the longitudinal displacement of each constituent defined by

$$u_1(X) = x_1(X) - X, \quad u_2(X) = x_2(X) - X, \tag{2.3}$$

and assume that u_1 and u_2 are small enough compared to l so that terms in u_1 and u_2 of orders greater than one are negligible.

If the gradients of u_1 and u_2 are also small, the corresponding strains have the forms

$$\epsilon_1 = \frac{\partial u_1}{\partial X} = u_1'(X), \quad \epsilon_2 = \frac{\partial u_2}{\partial X} = u_2'(X). \tag{2.4}$$

On assuming that each component is homogeneous, the stresses σ_1 and σ_2 are related to the strains by constitutive equations

$$\sigma_1 = E_1\epsilon_1 + E_3\epsilon_2, \quad \sigma_2 = E_2\epsilon_2 + E_4\epsilon_1, \tag{2.5}$$

where $E_1, \dots, E_4$ are elastic moduli. It is known (see Aron[3]) that these moduli must satisfy the conditions

$$E_1 > 0, \quad E_2 > 0, \quad E_1E_2 > \frac{1}{4}(E_3 + E_4)^2. \tag{2.6}$$

Since the constituents interact, let us denote by $m_1 = -m_2$ the interacting forces. These forces are not given in advance, but they depend on the kinematic variables through appropriate constitutive equations. We shall consider some particular forms of these constitutive equations in the following.

Stresses and interacting forces are related by balance equations. Neglecting all non-linear terms and ignoring the external body forces, these equations become

$$\frac{\partial \sigma_1}{\partial X} + m_1 = 0, \quad \frac{\partial \sigma_2}{\partial X} + m_2 = 0, \quad \text{in } 0 < X < l. \tag{2.7}$$

Using the constitutive equations (2.5) we obtain

$$\left.\begin{aligned} &E_1u_1'' + E_3u_2'' + m_1 = 0, \\ &E_2u_2'' + E_4u_1'' + m_2 = 0, \quad \text{in } 0 < X < l. \end{aligned}\right\} \tag{2.8}$$

As far the boundary conditions, we consider the bar placed in one of the two types of loading devices: the uniform device, in which the bar is fixed at one end and loaded at the other by a force P acting on both constituents; the single device, in which the bar is fixed at one end and loaded at the other by a force P acting only on one of the constituents.

3. THE PURELY ELASTIC BEHAVIOR

We first consider the simplest situation in which the interacting force $m_1 = -m_2$ is purely elastic. It is then known that a linear constitutive equation compatible with objectivity and isotropy has the form

$$m_1 = -m_2 = -a(u_1 - u_2), \tag{3.1}$$

where a is a constant ($a > 0$).

Substituting (3.1) into (2.8) we obtain

$$\left.\begin{aligned} E_1 u_1'' + E_3 u_2'' - a(u_1 - u_2) = 0, \\ E_2 u_2'' + E_4 u_1'' + a(u_1 - u_2) = 0 \quad \text{in } 0 < X < l. \end{aligned}\right\} \tag{3.2}$$

We now solve (3.2) in the case of uniform and single device.

With soft device the bar is fixed at one end and loaded at the other by a normal force P burdening on the two constituents (Fig. 1). The boundary conditions are

$$\left.\begin{aligned} u_1(0) = u_2(0) = 0, \\ u_1(l) = u_2(l), \\ \sigma_1(l) + \sigma_2(l) = \frac{P}{A}, \end{aligned}\right\} \tag{3.3}$$

where A is the cross section of the bar.

Since (3.2) is a system of ordinary differential equations with constant coefficients it is easy to find its general solution

$$\left.\begin{aligned} u_1 = B + CX + D(E_2 + E_3)\cosh \alpha X + E(E_2 + E_3)\sinh \alpha X, \\ u_2 = B + CX - D(E_1 + E_4)\cosh \alpha X - E(E_1 + E_4)\sinh \alpha X, \end{aligned}\right\} \tag{3.4}$$

where B, C, D, E are constants and α is given by

$$\alpha = \left[\frac{a(E_1 + E_2 + E_3 + E_4)}{E_1 E_2 - E_3 E_4}\right]^{1/2}, \tag{3.5}$$

and α is real since a is positive and (2.6) imply

$$E_1 + E_2 + E_3 + E_4 > 0,$$

$$4E_1E_2 > (E_3 + E_4)^2 = E_3^2 + 2E_3E_4 + E_4^2 \geq 4E_3E_4.$$

In order to determine B, C, D, E we use the boundary conditions (3.3). The first two of (3.3) give

$$B = D = 0,$$

and the others

$$E = 0,$$

$$C(E_1 + E_3) + C(E_2 + E_4) = \frac{P}{A},$$

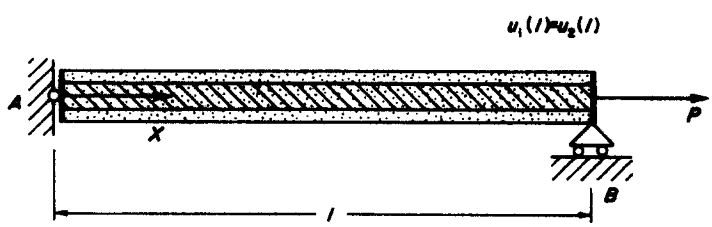

Fig. 1.

whence we derive

$$C = \frac{P}{(E_1 + E_2 + E_3 + E_4)A}.$$

We thus conclude that, in the bar, we have constant stresses of the type

$$\sigma_1 = \frac{(E_1 + E_3)P}{(E_1 + E_2 + E_3 + E_4)A}, \quad \sigma_2 = \frac{(E_2 + E_4)P}{(E_1 + E_2 + E_3 + E_4)A}, \tag{3.6}$$

and an interacting force $m_1 = -m_2$ vanishing everywhere.

We then consider the single device in which the total force at the end B (Fig. 2) is absorbed by the first constituent. The corresponding boundary conditions assumes the form

$$\left.\begin{aligned} u_1(0) = u_2(0) = 0, \\ \sigma_1(l) = \frac{P}{A}, \quad \sigma_2(l) = 0. \end{aligned}\right\} \tag{3.7}$$

Or using again the general solution (3.4) we determine the constants B, C, D, E with the new boundary conditions (3.7). An easy calculation yields

$$B = D = 0,$$

and

$$\begin{aligned} \sigma_1(l) = E_1 u_1'(l) + E_3 u_2'(l) = C(E_1 + E_3) \\ + \alpha \cosh \alpha l(E_1E_2 - E_3E_4)E = \frac{P}{A}, \end{aligned}$$

$$\begin{aligned} \sigma_2(l) = E_2 u_2'(l) + E_4 u_1'(l) = C(E_2 + E_4) \\ - \alpha \cosh \alpha l(E_1E_2 - E_3E_4)E = 0. \end{aligned}$$

Solving this system we obtain

$$C = \frac{P}{(E_1 + E_2 + E_3 + E_4)A}, \quad E = \frac{(E_2 + E_4)P}{\alpha \cosh \alpha l(E_1 + E_2 + E_3 + E_4)(E_1E_2 - E_3E_4)A}. \tag{3.8}$$

Once these constants are known, a simple substitution into (2.5) gives

$$\left.\begin{aligned} \sigma_1(X) = \frac{(E_1 + E_3)P}{(E_1 + E_2 + E_3 + E_4)A}\left[1 + \frac{\cosh \alpha X}{\cosh \alpha l}\frac{E_2 + E_4}{E_1 + E_3}\right], \\ \sigma_2(X) = \frac{(E_2 + E_4)P}{(E_1 + E_2 + E_3 + E_4)A}\left[1 - \frac{\cosh \alpha X}{\cosh \alpha l}e^{-\alpha(l-X)}\right], \end{aligned}\right\} \tag{3.9}$$

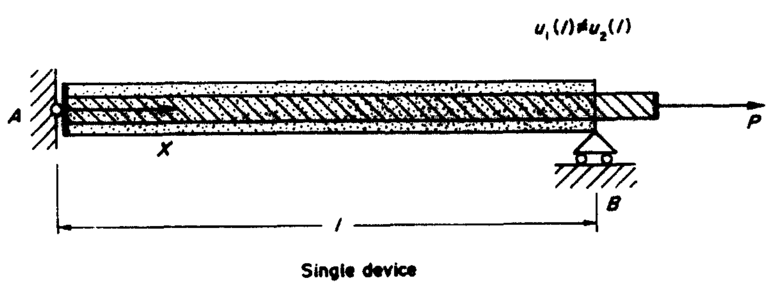

Fig. 2.

while the interacting force, derived from (3.1), is

$$m_1 = -m_2 = -\frac{a(E_2+E_4)P}{\alpha(E_1E_2-E_3E_4)A}\frac{\sinh \alpha X}{\cosh \alpha l}. \tag{3.10}$$

It is interesting to observe that $m_1 = -m_2$ attains its maximum in absolute value at the loaded end of the bar (Fig. 3). This distribution of the interacting force was also found by Hovgaard [7].

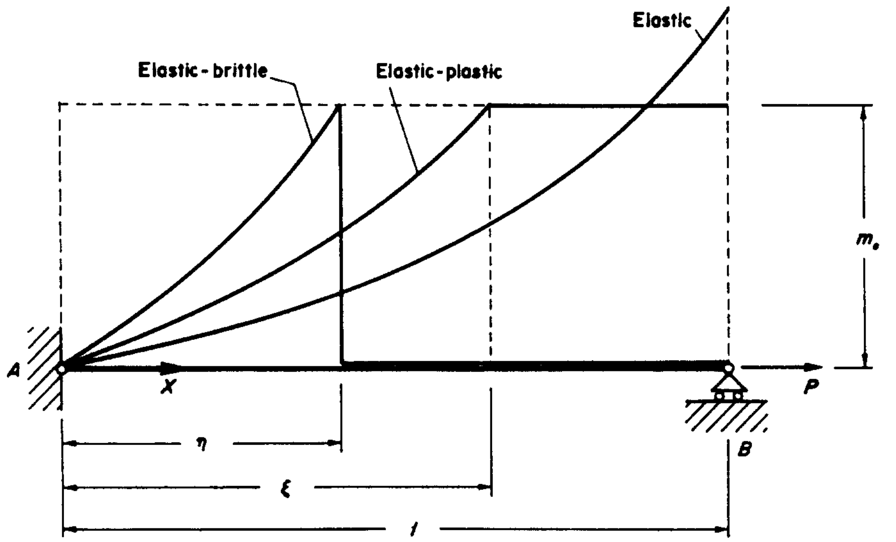

Fig. 3.

4. THE ELASTIC PLASTIC BEHAVIOR

Formula (3.10) shows that the interacting force tends to increase (in absolute value) close to one end. We wish now to introduce the additional assumption that this force satisfies a condition of the type

$$|m_1| = |m_2| \le m_0, \tag{4.1}$$

where m_0 is a prescribed value. The individual constituents are instead indefinitely elastic.

In case of uniform device the interacting force vanishes and therefore plasticisation cannot occur. We thus consider the case of single device and denote by $\xi (0<\xi<l)$ the point of transition between the elastic and the plastic region. Since m_1 is a monotone function it is reasonable to conjecture that this point is unique.

In the elastic region $0<X<\xi$ the solution still has the form (3.4). In the plastic region $\xi<X<l$, the interacting force has the value

$$m_1 = -m_2 = -m_0 = \text{const.}$$

and the equilibrium equations become

$$\left.\begin{aligned} &E_1u_1'' + E_3u_2'' - m_0 = 0, \\ &E_2u_2'' + E_4u_1'' + m_0 = 0 \ \text{ in } \xi < X < l \end{aligned}\right\}. \tag{4.2}$$

The general solution of this system is

$$\left.\begin{aligned} u_1 &= F + GX + \frac{m_0}{2}\frac{E_2+E_3}{E_1E_2-E_3E_4}X^2, \\ u_2 &= H + IX - \frac{m_0}{2}\frac{E_1+E_4}{E_1E_2-E_3E_4}X^2 \ \text{ in } \xi<X<l, \end{aligned}\right\} \tag{4.3}$$

where F, G, H, I are constants.

The eight constants B, C, D, E, F, G, H, I and the position ξ must be determined by the boundary conditions

$$u_1(0) = u_2(0) = 0; \; \sigma_1(l) = \frac{P}{A}, \; \sigma_2(l) = 0, \tag{4.4}$$

and the continuity conditions at ξ

$$\left.\begin{aligned} u_1(\xi^+) &= u_1(\xi^-), \\ u_2(\xi^+) &= u_2(\xi^-), \\ \sigma_1(\xi^+) &= \sigma_1(\xi^-), \\ \sigma_2(\xi^+) &= \sigma_2(\xi^-), \\ m_1(\xi^-) &= -m_0. \end{aligned}\right\} \tag{4.5}$$

Since from (4.3) and (2.5) we have

$$\left.\begin{aligned} \sigma_1(X) &= E_1\Big(G + m_0\frac{E_2+E_3}{E_1E_2-E_3E_4}X\Big) + E_3\Big(I - m_0\frac{E_1+E_4}{E_1E_2-E_3E_4}X\Big), \\ \sigma_2(X) &= E_2\Big(I - m_0\frac{E_1+E_4}{E_1E_2-E_3E_4}X\Big) + E_4\Big(G + m_0\frac{E_2+E_3}{E_1E_2-E_3E_4}X\Big) \text{ in } \xi < X < l, \end{aligned}\right\} \tag{4.6}$$

we can write (4.4) by using (3.4) and (4.6). A long, but easy calculation, enables us to obtain

$$\left.\begin{aligned} B &= D = 0, \\ G &= -\frac{E_2+E_3}{E_1E_2-E_3E_4}m_0 l + \frac{E_2}{E_1E_2-E_3E_4}\frac{P}{A}, \\ I &= \frac{E_1+E_4}{E_1E_2-E_3E_4}m_0 l - \frac{E_4}{E_1E_2-E_3E_4}\frac{P}{A}, \end{aligned}\right\} \tag{4.7}$$

and the substitution of (4.7) into (4.6) yields

$$\left.\begin{aligned} \sigma_1(X) &= -m_0(l-X) + \frac{P}{A}, \\ \sigma_2(X) &= m_0(l-X) \text{ in } \xi < X < l. \end{aligned}\right\} \tag{4.8}$$

The continuity conditions (4.5) then become

$$\left.\begin{aligned} C\xi + E(E_2+E_3)\sinh\alpha\xi &= F + G\xi + \frac{m_0}{2}\frac{E_2+E_3}{E_1E_2-E_3E_4}\xi^2, \\ C\xi - E(E_1+E_4)\sinh\alpha\xi &= H + I\xi - \frac{m_0}{2}\frac{E_1+E_4}{E_1E_2-E_3E_4}\xi^2, \\ C(E_1+E_3) + E\alpha\cosh\alpha\xi(E_1E_2-E_3E_4) &= -m_0(l-\xi) + \frac{P}{A}, \\ C(E_2+E_4) - E\alpha\cosh\alpha\xi(E_1E_2-E_3E_4) &= m_0(l-\xi), \\ aE(E_1+E_2+E_3+E_4)\sinh\alpha\xi &= m_0. \end{aligned}\right\} \tag{4.9}$$

In order to solve (4.9) we suppose for the moment of having found a value of ξ such that (4.9) is compatible.

Once ξ is known, we obtain C and E from $(4.9)_3$, $(4.9)_4$:

$$C=\frac{P}{(E_1+E_2+E_3+E_4)A},$$

$$E=\frac{1}{\alpha\cosh\alpha\xi(E_1E_2-E_3E_4)}\left[\frac{(E_2+E_4)P}{(E_1+E_2+E_3+E_4)A}-m_0(l-\xi)\right];$$

and from $(4.9)_1$, $(4.9)_2$ we calculate

$$F=\left[\xi+\frac{\tanh\alpha\xi(E_2+E_3)(E_2+E_4)}{\alpha(E_1E_2-E_3E_4)}-\frac{\xi E_2(E_1+E_2+E_3+E_4)}{E_1E_2-E_3E_4}\right]$$
$$\times\frac{P}{(E_1+E_2+E_3+E_4)A}-\frac{m_0(E_2+E_3)}{E_1E_2-E_3E_4}\left[\frac{1}{\alpha}(l-\xi)\tanh\alpha\xi-l\xi+\frac{1}{2}\xi^2\right],$$

$$H=\left[\xi-\frac{\tanh\alpha\xi(E_2+E_4)(E_1+E_4)}{\alpha(E_1E_2-E_3E_4)}+\frac{\xi E_2(E_1+E_2+E_3+E_4)}{E_1E_2-E_3E_4}\right]$$
$$\times\frac{P}{(E_1+E_2+E_3+E_4)A}+\frac{m_0(E_1+E_4)}{E_1E_2-E_3E_4}\left[\frac{1}{\alpha}(l-\xi)\tanh\alpha\xi-l\xi+\frac{1}{2}\xi^2\right].$$

With these values of the constants we find that the interacting force is

$$m_1=-m_2=\begin{cases}-\dfrac{a(E_1+E_2+E_3+E_4)}{\alpha\cosh\alpha\xi(E_1E_2-E_3E_4)}\left[\dfrac{(E_2+E_4)P}{(E_1+E_2+E_3+E_4)A}-m_0(l-\xi)\right]\sinh\alpha X & \text{in } 0\le X<\xi,\\ -m_0 \text{ in } \xi\le X\le l.\end{cases}$$

It only remains to evaluate ξ from the condition of continuity

$$\frac{a(E_1+E_2+E_3+E_4)}{\alpha\cosh\alpha\xi(E_1E_2-E_3E_4)}\left[\frac{(E_2+E_4)P}{(E_1+E_2+E_3+E_4)A}-m_0(l-\xi)\right]\sinh\alpha\xi=m_0.$$

On rewriting this equations in the form

$$\tanh\alpha\xi=\frac{m_0\alpha(E_1E_2-E_3E_4)}{a(E_1+E_2+E_3+E_4)}\frac{1}{\dfrac{(E_2+E_4)P}{(E_1+E_2+E_3+E_4)A}-m_0l+m_0\xi}, \tag{4.10}$$

we immediately see that it admits only one positive solution, which confirms our conjecture on the uniqueness of the point of transition. It may be interesting to observe that, if, for $\xi=l$ in (4.10), we find

$$\tanh\alpha l<\frac{m_0\alpha(E_1E_2-E_3E_4)A}{a(E_2+E_4)P},$$

the bar does not plasticise at any point and the solution is like (3.4).

The diagram of $-m_1(X)$ is qualitatively represented in Fig. 3.

5. THE ELASTIC-BRITTLE BEHAVIOR

Let us now assume that the interacting force exhibits a brittle behavior of the following type. It is purely elastic as far as $m_1=-m_2$, in absolute value, is not greater than a given value m_0, but it falls suddenly to zero at the points at which it would exceed, in absolute value, the bound m_0. This behavior can be mathematically described by the following equation

$$m_1=-m_2=\begin{cases}-a(u_1-u_2) & \text{for } |a(u_1-u_2)|\le m_0,\\ 0 & \text{for } |a(u_1-u_2)|>m_0\end{cases}. \tag{5.1}$$

Since in the case of uniform loading device the interacting force is identically zero, we only consider the single device and denote by $\eta(0 \le \eta \le l)$ the point of transition between the elastic and the non-reacting region.

In this region the equilibrium equations are

$$\left.\begin{aligned} &E_1 u_1'' + E_3 u_2'' = 0, \\ &E_2 u_2'' + E_4 u_1'' = 0 \text{ in } \eta < X < l \end{aligned}\right\}, \tag{5.2}$$

and the corresponding solution is

$$\left.\begin{aligned} &u_1 = K + LX, \\ &u_2 = M + NX \text{ in } \eta < X < l \end{aligned}\right\}, \tag{5.3}$$

where K, L, M, N are constants.

Since in the elastic region $0 \le X < \eta$ the solution is of the form (3.4) we must determine the eight constants B, C, D, E, K, L, M, N by the boundary conditions (4.4) and the continuity conditions, which are like (4.5) with ξ replaced by η.

On using (5.3) and (2.5) we obtain

$$\left.\begin{aligned} &\sigma_1(X) = E_1 L + E_3 N, \\ &\sigma_2(X) = E_2 N + E_4 L \text{ in } \eta < X < l \end{aligned}\right\}. \tag{5.4}$$

The $(4.4)_1$, $(4.4)_2$ give

$$B = D = 0,$$

and $(4.4)_3$, $(4.4)_4$ yield

$$\left.\begin{aligned} &E_1 L + E_3 N = \frac{P}{A}, \\ &E_2 N + E_4 L = 0, \end{aligned}\right\}$$

whence we derive

$$L = \frac{E_2 P}{(E_1E_2 - E_3E_4)A}, \quad N = -\frac{E_4 P}{(E_1E_2 - E_3E_4)A}, \tag{5.5}$$

that is,

$$\sigma_1(X) = \frac{P}{A}, \quad \sigma_2(X) = 0 \text{ in } \eta < X < l. \tag{5.6}$$

The continuity conditions assume the form

$$\left.\begin{aligned} &C\eta + E(E_2 + E_3)\sinh \alpha\eta = K + L\eta, \\ &C\eta - E(E_1 + E_4)\sinh \alpha\eta = M + N\eta, \\ &C(E_1 + E_3) + E\alpha \cosh \alpha\eta(E_1E_2 - E_3E_4) = \frac{P}{A}, \\ &C(E_2 + E_4) - E\alpha \cosh \alpha\eta(E_1E_2 - E_3E_4) = 0, \\ &aE(E_1 + E_2 + E_3 + E_4)\sinh \alpha\eta = m_0. \end{aligned}\right\} \tag{5.7}$$

Let us suppose, as before, that there is an η for which (5.7) is compatible. From $(5.7)_3$, $(5.7)_4$ we find C and E:

$$C = \frac{P}{(E_1+E_2+E_3+E_4)A},$$

$$E = \frac{(E_2+E_4)P}{\alpha \cosh \alpha\eta (E_1E_2 - E_3E_4)(E_1+E_2+E_3+E_4)A},$$

and from $(5.7)_1$, $(5.7)_2$ we have:

$$K = \left[\eta + \frac{\tanh \alpha\eta (E_2+E_3)(E_2+E_4)}{\alpha(E_1E_2-E_3E_4)} - \frac{\eta E_2(E_1+E_2+E_3+E_4)}{E_1E_2-E_3E_4}\right] \times \frac{P}{(E_1+E_2+E_3+E_4)A},$$

$$L = \left[\eta - \frac{\tanh \alpha\eta (E_2+E_4)(E_1+E_4)}{(1+\alpha\eta)(E_1E_2-E_3E_4)} + \frac{\eta E_2(E_1+E_2+E_3+E_4)}{E_1E_2-E_3E_4}\right] \times \frac{P}{(E_1+E_2+E_3+E_4)A}.$$

The interacting force is

$$m_1 = -m_2 = \begin{cases} -\dfrac{a(E_2+E_4)P}{\alpha \cosh \alpha\eta (E_1E_2-E_3E_4)} \sinh \alpha X \text{ in } 0 \le X \le \eta, \\ 0 \text{ in } \eta < X \le l. \end{cases}$$

In order to evaluate η we use $(5.7)_5$:

$$\frac{a(E_2+E_4)P \sinh \alpha\eta}{\alpha \cosh \alpha\eta (E_1E_2-E_3E_4)A} = m_0.$$

This equation can be written as

$$\tanh \alpha\eta = \frac{\alpha m_0(E_1E_2-E_3E_4)A}{a(E_2+E_4)P}.$$

Since, by virtue of the constitutive assumptions, the r.h.s. is positive, the only possible solution is positive. If, however, it happens that

$$\tanh \alpha\eta < \frac{\alpha m_0(E_1E_2-E_3E_4)A}{a(E_2+E_4)P},$$

the transition point falls beyond l and the bar remains elastic.

The diagram of $-m_1(X)$ is reproduced in Fig. 3.

6. FORCE-DISPLACEMENT DIAGRAMS

The three types of behavior studied above can be expressively compared by examining the dependence of the external load P, on the displacement $u_1(l)$ of the first contituent at the end B (Fig. 2).

If the interacting force is indefinitely elastic, we can find $u_1(l)$ from $(3.4)_1$. Using (3.8) and solving $(3.4)_1$ with respect to P we obtain

$$P = \frac{(E_1+E_2+E_3+E_4)Au_1(l)}{\left[1 + \dfrac{\tanh \alpha l (E_2+E_4)(E_2+E_3)}{\alpha(E_1E_2-E_3E_4)}\right]}. \tag{6.1}$$

This equation represents a straight line and its diagram is represented in Fig. 4.

When the interacting force behaves plastically, the relationship between P and $u_1(l)$ is given by $(4.3)_1$. On giving the constants their values and rearranging the terms, we obtain

$$P = \frac{(E_1+E_2+E_3-E_4)Au_1(l)}{\left[l + \left(\dfrac{1}{\alpha}\tanh \alpha\xi + l - \xi\right)\dfrac{(E_2+E_4)(E_2+E_3)}{E_1E_2-E_3E_4}\right]} + \frac{m_0(E_2+E_3)}{2(E_1E_2-E_3E_4)} \times (l-\xi)^2. \tag{6.2}$$

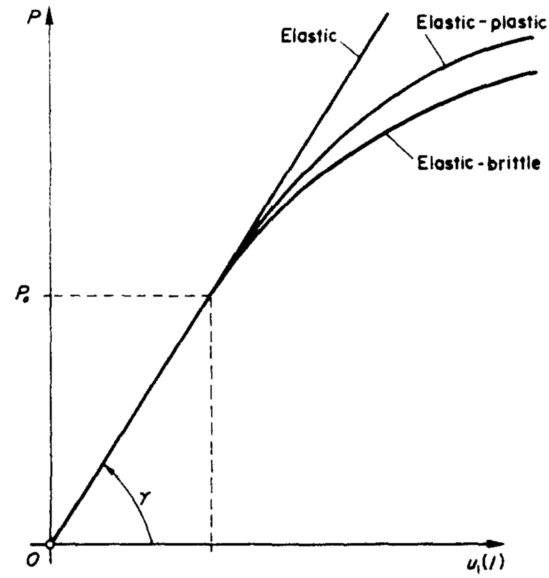

Fig. 4.

In this equation, for $P < Am_0\alpha(E_1E_2 - E_3E_4A)/\alpha(E_2 + E_4)\tanh \alpha l A = P_0$, ξ is identically equal to l and the bar behaves elastically. When, instead, P overcomes the point of first plasticisation the diagram is no longer linear since ξ depends on P. For P ranging from $P = P_0$ to infinity, ξ describes monotonically the interval $l \geq \xi > 0$. The graph of (6.2) is qualitatively represented in Fig. 4.

Finally, consider the brittle behavior. Performing the customary substitutions on $(5.3)_1$ we obtain

$$P = \frac{(E_1 + E_2 + E_3 + E_4)Au_1(l)}{\left[l + \left(\frac{1}{\alpha}\tanh \alpha\eta + l - \eta\right)\frac{(E_2 + E_4)(E_2 + E_3)}{E_1E_2 - E_3E_4}\right]}, \tag{6.3}$$

where $\eta = l$ for $P < P_0$. When P, overcoming this value, tends to infinity the value of η varies monotonically from l to zero. The corresponding graph, represented in Fig. 4, remains under the curve (6.2).

However, an unexpected property is that in the limit, for ξ and η tending to zero, both these curves have the same derivative

$$\lim_{P\to\infty} \frac{dP}{du_1(l)} = \frac{(E_1 + E_2 + E_3 + E_4)A}{l\left[1 + \frac{(E_2 + E_4)(E_2 + E_3)}{E_1E_2 - E_3E_4}\right]}. \tag{6.4}$$

This derivative is clearly smaller than the ratio $P/u_1(l)$ of the elastic case.

REFERENCES

1. A. E. Green and P. M. Naghdi, A dynamical theory of interacting continua. *Int. J. Engng Sci.* **3**, 231–241 (1965).
2. M. J. Crochet and P. M. Naghdi, On constitutive equations for flow of fluid through an elastic solid. *Int. J. Engng Sci.* **4**, 383–401 (1966).
3. A. E. Green and T. R. Steel, Constitutive equations for interacting continua. *Int. J. Engng Sci.* **4**, 483–500 (1966).
4. M. Aron, On the plane strain problem for an isotropic mixture of two linear elastic solids *Arch. Mech. Stosow.* **27**(4), 679–684 (1975).
5. H. F. Tiersten and M. Jahanmir, A theory of composites modeled as interpenetrating solid continua. *Arch. Rat. Mech. Anal.* **65**, 153–192 (1977).
6. M. Jahanmir and H. F. Tiersten, Load transfer and surface wave propagation in fiber reinforced composite materials. *Int. J. Solids Structures* **14**, 227–240 (1978).
7. W. Hovgaard, Die spannungsverteilung in Schweissungen. *Zeit. Angew. Math. Mech.* **11**(5), 341–348 (1931).